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AKTS.

Stabilization of a genuine G-topos.

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Recall: stability, stabilization.

$\mathcal{C}$ pointed, presentable ∞-cat. E.g. S_*.

Stable ∞-cat:

- pointed,
- finite colimits,
-

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ W \arrow[r] & Z \end{tikzcd}$$

pushout ⇔ pullback.

Given $\mathcal{C}$ pointed, can consider functor $\Sigma : \mathcal{C} \to \mathcal{C}$.

Form $\operatorname{colim}(\mathcal{C} \xrightarrow{\Sigma} \mathcal{C} \xrightarrow{\Sigma} \mathcal{C} \xrightarrow{\Sigma} \cdots) = Sp(\mathcal{C})$.

$$Sp(S_*) \simeq Sp.$$

Given G-spaces $\operatorname{Fun}(\mathrm{Orb}_G^{op}, S_*)$,

the stabilization is naive G-spectra.

Genuine G-spectra.

- start with naive and invert representation spheres S^V.
- Spectral Mackey functors

$$Sp^G = \operatorname{Fun}^{\times}(\operatorname{Span}(\mathbb{F}_G), Sp).$$

Given a "G-topos", is there a genuine notion of stabilization?

Input data. G a finite group.

$\mathrm{Fun}(BG, \mathrm{RTop}_\infty)$.

$\uparrow$

Objects are ∞-topoi (lex loc. of $\mathcal{P}(\mathcal{C})$). **Ex.** $\mathfrak{X} = \mathrm{Shv}(X)$ $\uparrow$ topological space X.

$$\begin{tikzcd} \mathfrak{X} \arrow[r, shift right, "\pi_*"'] & \mathfrak{Y} \arrow[l, hook', shift right, "\pi^*"'] \end{tikzcd}$$
, π^* is lex.

Directionality is π_*.

Ex. If X has a G-action,

$$\mathrm{Shv}(X)^{BG}.$$

Same for $\mathcal{S}_{/X}$.

Goal: describe stable ∞-cat $\mathrm{Sp}^G(\mathfrak{X})$ s.t. $\mathrm{Sp}^G(\overset{\text{trivial action}}{\mathcal{S}}) \simeq \mathrm{Sp}^G$.

Second motivation. Scheiderer's work on real étale cohomology.

X a scheme.

$\mathrm{Et}_{/X}$ étale morphisms $U \to X$.

Make into a site. Étale topos

$$\tilde{X}_{\text{ét}} = \mathrm{Shv}^{\text{ét}}(\mathrm{Et}_{/X}).$$

We *also* have the real étale topology on $\mathrm{Et}_{/X}$.

Given a commutative ring A, get

$$\underset{\substack{\| \\ \mathrm{RSpec}}}{\mathrm{Sper}(A)} = \{ (\mathfrak{p}, \alpha) : \mathfrak{p} \in \mathrm{Spec}(A),\ \alpha \text{ an ordering on } \kappa(\mathfrak{p}) \}.$$

Extend to schemes.

New topology on $Et_{/X}$ where covers are jointly surjective on Sper.

$\widetilde{X}_{rét}$. In fact, $\widetilde{X}_{rét} \simeq Shv(Sper\ X)$.

Ex. $X = Spec\ \mathbb{R}$.

$\widetilde{X}_{ét} = C_2\text{-sets}$.

$\widetilde{X}_{ret} = Sets$.

Ex. $X = Spec\ k$.

$\Gamma = Gal(k^s/k)$.

$T \subseteq \Gamma$ subset of $\overset{\text{non-zero}}{\text{order}}$ 2 elements.

Γ acts on T by conjugation.

$\mathbb{R}Spec(k) = T/\Gamma$

$\widetilde{X}_{rét} = Shv(T/\Gamma)$.

Ex. k a # field,

$T/\Gamma \cong \{\text{real places}\}$.

Rem. X noetherian sch of f. dim

$SH(X)$

$\rho : \mathbb{S}^{0,0} \longrightarrow \mathbb{G}_m$ corresponds to -1 in $\mathbb{G}_m(X)$.

$SH(X)[\rho^{-1}] \underset{\text{Bachmann}}{\simeq} Sp(Shv^{ret}(X))$.

Grothendieck topology "b"

$b = \acute{e}t \cap ret.$ "Biggest topology coarser than both."

$$\tilde{X}_b = \mathcal{P}_{Set}(\mathcal{O}_{C_2})$$

$X = \mathrm{Spec}\, \mathbb{R}.$

Colimits in $RTop_\infty$.

$$RTop_\infty^{op} \simeq LTop_\infty \subseteq Pr^L$$

colimits are limits computed in $\widehat{Cat}_\infty$.

Grothendieck construction.

Cartesian sections.

$k = BG$

$\mathfrak{X}$ G-topos

Grothendieck construct.

$$\mathfrak{X}_{hG} = \mathrm{Fun}_{/BG}(BG, \underline{\mathfrak{X}})$$

If G acts trivially,

$$\mathfrak{X}_{hG} = \mathrm{Fun}_{/BG}(BG, BG \times \mathfrak{X}) \simeq \mathrm{Fun}(BG, \mathfrak{X}).$$

Limits in RTop are generally hard.

Ex. X G-space.

$\mathfrak{X} = \mathrm{Shv}(X)$.

$$\mathrm{Shv}(X)^{hG} \underset{\uparrow}{=} \mathrm{Shv}(X^G)$$

G acts prop. and disc. on X.

Ex. C_2-topos.

S a scheme

$S[\underset{\zeta_2}{\overset{\shortparallel}{i}}] \circlearrowleft C_2$

$\mathfrak{X} = \widetilde{S[i]}_{\text{ét}} \circlearrowleft C_2$

$\mathfrak{X}^{hC_2} = \widetilde{S}_{\text{ret}}$

$\mathfrak{X}_{hC_2} = \widetilde{S}_{\text{ét}}$.

2 invertible on S

$G = C_2$

$\mathfrak{X}$ C_2-topos

G-equivariant for trivial C_2-action on $\mathfrak{X}^{hC_2}$.

$$\begin{tikzcd}
\mathfrak{X} \arrow[r, "\iota^*"] \arrow[d, "\pi_*"'] & \mathfrak{X}^{hC_2} \\
\mathfrak{X}_{hC_2} \arrow[ur, dashed, "p"] \arrow[r, "V(\mathcal{E}_2)^+"'] & \mathrm{Fun}(BC_2, \mathfrak{X}^{hC_2}) \arrow[u, "(-)^{hC_2}"']
\end{tikzcd}$$

Recollements.

Idea: glue $\mathcal{X}_{hC_2}, \mathcal{X}^{hC_2}$ via the topos norm p.

$$\begin{tikzcd} \mathcal{U} \arrow[r, shift right, "j_*"'] \arrow[rr, bend left, "i^*j_*"] & \mathcal{X} \arrow[l, shift right, "j^*"'] \arrow[r, shift right, "i_*"'] & \mathcal{Z} \arrow[l, shift right, "i^*"'] \end{tikzcd}.$$

$$i^*X \to i^*j_*j^*X.$$

$$Shv(U) \rightleftarrows Shv(X) \rightleftarrows Shv(Z)$$

$$Z = X \setminus U.$$

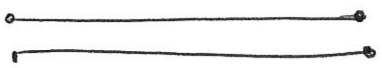
Need gluing functor to be lex.

Def. "Quotient topos" $\mathcal{X}/C_2$ is glue $\mathcal{X}_{hC_2}, \mathcal{X}^{hC_2}$ along p.

$$\mathcal{X} = \mathcal{S}_{\blacksquare}.$$

$$Fun(BG, \mathcal{S}) \xrightarrow{(-)^{hC_2}} \mathcal{S}.$$

$$\mathcal{X}/C_2 = \mathcal{P}(\mathcal{O}_{C_2}).$$



Stabilize fiberwise.

$$Sp(\mathcal{X}/C_2) \xrightarrow[res]{} Sp(\mathcal{X})^{\circlearrowleft C_2}$$

Then, force the left and right adjoints to coincide.

Whats the theory of stabilization?

Input: presheaf of cats on $\mathcal{O}_G$.

$$\mathcal{X}/C_2 \to \mathcal{X}^{\circlearrowleft C_2}$$

Thm. Yields $Sp^G(\mathcal{X})$.

Explicit description by G-commutative monoids.

For $\widetilde{S[i]}_{\acute{e}t}/C_2 \simeq \tilde{S}_b$.

Back to b.